Problem Set 3 (Solutions by Joey Heerens)

Problem 1 *Let $D(n)$ be the number of derangements in S_n*

1. *Prove that $D(n) = (n-1)(D(n-1) + D(n-2))$.*
2. *Deduce that $D(n) = nD(n-1) + (-1)^n$.*

Solution. Let D_n be the set of derangements of $[n]$. Note that for any permutation $\pi \in D_n$, the element 1 can not be in a self cycle, so there are $n-1$ choices for $\pi(1) = k$. There are two cases we now consider for $\pi(k)$. The first is if $\pi(k) = 1$, in which we have to arrange the other $n-2$ elements such that for no i is $\pi(i) = i$. This is easily equivalent to $D(n-2)$.

The other case is if $\pi(k) \neq 1$. Let the set of derangements with $\pi(\pi(1)) \neq 1$ be denoted A_n. We claim there is a bijection $f : A_n \to D_{n-1}$. For each $\pi \in A_n$, we transform π into a $\pi' \in D_{n-1}$ by letting $\pi'(1) = \ell$ where $\pi(k) = \ell$, and ignoring the element k in π'. This gives us $n-1$ elements that must not be in self-cycles, and this process is easily reversible. Therefore, f is a bijection.

We have shown that $D(n) = (n-1)D(n-1)+(n-1)D(n-2)$, proving part 1. Now, part 2 can be shown through induction. $D(1) = 0$ and $D(2) = 1$, so the base case holds. Assume that the hypothesis is true for some $k = n$. Then, $D(k+1) = k(D(k)+D(k-1))$ from part 1. Further,

$$D(k+1)-(k+1)D(k) = kD(k-1)-D(k) = kD(k-1)-kD(k-1)-(-1)^k = (-1)^{k+1}$$

from the induction hypothesis. This means $D(k+1) = (k+1)D(k) + (-1)^{k+1}$, so the induction holds and we are done. □

Problem 2 *For each $n \in \mathbb{N}_0$, let C_n be the n-th Catalan number and set $a_n = nC_n$. Find an explicit formula for the generating function of $(a_n)_{n\geq 0}$.*

Solution. Recall from lecture that the generating function for $(C_n)_{n\geq 0}$ is $\frac{1-\sqrt{1-4x}}{2x}$. Further, if $C(x)$ is the generating function for $(C_n)_{n\geq 0}$, then

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} nC_n x^n = x\sum_{n=0}^{\infty} nC_n x^{n-1} = xC'(x).$$

From quotient rule, the derivative of $C(x)$ is

$$\frac{-2x-\sqrt{1-4x}+1}{2x^2\sqrt{1-4x}}.$$

Thus, the generating function is $\frac{-2x-\sqrt{1-4x}+1}{2x\sqrt{1-4x}}$. □

Problem 3 *Find an explicit formula for the number of solutions* $(x, y, z) \in \mathbb{N}_0^3$ *of the equation* $x + y + z = n$ *satisfying that* x *is odd,* $y > 2$*, and* $z < 5$.

Solution. Let the number of solutions to $x + y + z = n$ for $(x, y, z) \in \mathbb{N}_0^3$ be a_n. If $A(x)$ is the generating function for $(a_n)_{n\geq 0}$, then

$$A(x) = (x + x^3 + ...)(x^3 + x^4 + ...)(1 + x + x^2 + x^3 + x^4) = \frac{x}{1 - x^2} \cdot \frac{x^3}{1 - x} \cdot \frac{1 - x^5}{1 - x},$$

where we get the product of function as each exponent corresponds to the potential choices for x, y, and z. It remains to find an explicit form for each coefficient of $A(x)$. It is known that

$$\frac{1}{(1 - x)^2} = 1 + 2x + 3x^2 + ... = \sum_{n=0}^{\infty}(n + 1)x^n.$$

Thus,

$$A(x) = \left(\sum_{n=0}^{\infty}(n + 1)x^n\right)\left(\sum_{n=0}^{\infty} x^{2n}\right)(x^4(1 - x^5)).$$

To find the x^k coefficient of the product of the first two infinite sums, we can look at each 2ℓ with $\ell \leq \lfloor \frac{k}{2} \rfloor$ in the right sum. This implies the coefficient of x^k is

$$\sum_{n=0}^{\lfloor \frac{k}{2} \rfloor}(k-2n+1) = (k+1)\left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) - \left\lfloor \frac{k}{2} \right\rfloor\left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) = \left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right)\left(\left\lfloor \frac{k + 1}{2} \right\rfloor + 1\right).$$

From here, the a_n can be evaluate as

$$\left(\left\lfloor \frac{n - 4}{2} \right\rfloor + 1\right)\left(\left\lfloor \frac{n - 3}{2} \right\rfloor + 1\right) - \left(\left\lfloor \frac{n - 9}{2} \right\rfloor + 1\right)\left(\left\lfloor \frac{n - 8}{2} \right\rfloor + 1\right) = \left\lfloor \frac{5}{2}n \right\rfloor - 11.$$

This is the explicit form for the number of solutions when $n > 5$, as the last expression will be negative when $n \leq 5$. To cover the other cases, note that there is 1 solution when $n = 4$, and 2 solutions when $n = 5$. This covers all general solutions. □

Problem 4 *Let* a_n *be the number of compositions of* n *with an odd number of parts such that every part is at least* 3. *Find an explicit formula (no summation signs allowed) for the generating function of* $(a_n)_{n\geq 0}$.

Solution. Let b_n be the sequence that represents partitioning n into odd parts and c_n be the sequence representing that each part is at least 3. Further, $B(x)$ is the generating function for $(b_n)_{n\geq 0}$ and $C(x)$ is the generating function for $(c_n)_{n\geq 0}$. From the composition theorem, $(a_n)_{n\geq 0}$ is simply $B(C(x))$. However, we know that

$$B(x) = \sum_{k \text{ odd}} x^k = \frac{x}{1-x^2}$$

and

$$C(x) = \sum_{n=3}^{\infty} x^n = \frac{x^3}{1-x}.$$

Therefore, it can be seen that

$$A(x) = \frac{\frac{x^3}{1-x}}{1-\left(\frac{x^3}{1-x}\right)^2} = \frac{x^3(1-x)}{1-2x+x^2-x^6}.$$

□

Problem 5 *Let t_n be the number of partitions of $[n]$ into blocks of cardinality two. Find the explicit formula (no summation signs allowed) for the exponential generating function of $(t_n)_{n\geq 0}$.*

Solution. Let a_n be the sequence corresponding to do nothing to an n element set, and b_n the sequence corresponding to taking a set of cardinality two. Clearly, $a_n = 1$ for all n and $b_n = 0$ at all points except for $n = 2$, where $b_2 = 1$. This means the exponential generating function for $A(x) = (a_n)_{n\geq 0}$ is e^x and the exponential generating function for $B(x) = (b_n)_{n\geq 0}$ is $\frac{x^2}{2}$. The composition theorem tells us that $(t_n)_{n\geq 0}$ can be represented as $A(B(x)) = e^{x^2/2}$, which is the answer. □

Problem 6 *Find an explicit formula (no summation signs allowed) for the exponential generating function of $(D(n))_{n\geq 0}$, where $D(0) = 1$ and $D(n)$ is the number of derangements of S_n.*

Solution. Recall from problem 1 that $D(n) = nD(n-1) + (-1)^n$. Let the exponential generating function for $(D(n))_{n\geq 0}$ be $F(x)$. Then we see

$$\begin{aligned} F(x) &= \sum_{n=0}^{\infty} \frac{D(n)}{n!} x^n = \sum_{n=0}^{\infty} \frac{nD(n-1)}{n!} x^n + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \\ &= x \sum_{n=1}^{\infty} \frac{D(n-1)}{(n-1)!} x^{n-1} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = xF(x) + e^{-x}. \end{aligned}$$

Thus, it can be deduced that

$$(1-x)F(x) = e^{-x} \implies F(x) = \frac{e^{-x}}{1-x}.$$

□

Problem 7 *For each $n \in \mathbb{N}$, let t_n be the number of simple graphs with vertex set $[n]$ with no vertex of degree larger than 2, and assume that $t_0 = 1$. Find an explicit formula for the exponential generating function of $(t_n)_{n\geq 0}$.*

Solution. We proposed the following lemma.

Lemma 8 *If every vertex in a connected component has a degree at most 2, then this connected component is either of the form C_n or P_n. (C_n is a cycle on n vertices that looks like a polygon, and P_n is a chain of n vertices with the two endpoints having degree 1 and the rest having degree 2)*

Proof of Lemma. We prove the lemma via induction. First assume that there exists a vertex of degree 1, and we claim that the only possible graph is P_n. The base case is clearly true for $n = 1$ and $n = 2$. Now assume the inductive hypothesis is true for some $k = n$. Then, if there is a graph on $k+1$ vertices where some vertex v has degree 1, say G, then the graph $G' = G\backslash\{v\}$ is a graph on k vertices, and the vertex adjacent to v must have degree either 1 or 0 in the graph G'. If it were 0, then the connect component would make P_2. Otherwise, we can use the inductive hypothesis to see that the graph $G' = P_k$, meaning that $G = P_{k+1}$. This proves the inductive hypothesis.

Otherwise, every vertex has degree 2. This means the sum of all degrees are $2n$, meaning that there are n edges and thus this graph can't be a tree. Then we know there exists a cycle in the graph, and further if any of the vertices in this cycle are connected to a vertex outside the cycle, then the degree of that vertex would be greater than 2. Therefore, every vertex in our graph must be apart of the same cycle, showing that it is C_n. Thus, the lemma follows.

Using the lemma, we can find the exponential generating function. The first thing we know is that there are $\frac{(n-1)!}{2}$ different possible graphs for C_n and $\frac{n!}{2}$ different possible graphs for P_n. Let s_n be the number of simple vertex sets in $[n]$ subject to the problem conditions given a single connected component. Thus, $s_n = \frac{(n-1)!+n!}{2}$ for $n \geq 3$. If $n = 2$, then $s_n = 1$ since there is no 2-cycle and also if $n = 1$, then $s_n = 1$. Now let

$F(x)$ be the generating function for $(s_n)_{n\geq 0}$. Therefore

$$\begin{aligned} F(x) &= x + \frac{x^2}{2} + \sum_{n=3}^{\infty} \frac{(n-1)! + n!}{2 \cdot n!} x^n \\ &= x + \frac{x^2}{2} + \sum_{n=3}^{\infty} \left(\frac{1}{2} + \frac{1}{2n} \right) x^n \\ &= -\frac{x^2}{4} + \sum_{n=1}^{\infty} \left(\frac{1}{2} + \frac{1}{2n} \right) x^n. \end{aligned}$$

We can then compute each part of the sum. It is known that

$$\sum_{n=1}^{\infty} \frac{x^n}{2} = \frac{x}{2(1-x)}$$

and that

$$\sum_{n=1}^{\infty} \frac{x^n}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{x^n}{n} = \frac{1}{2} \ln\left(\frac{1}{1-x} \right).$$

The last equality comes from the integral of $\frac{1}{1-x}$.

The only consideration left is that there can be multiple connected components. In this case, the exponential generating function to do nothing with n connected components is $G(x)$, which is simply e^x. Therefore, from the composition theorem, the generating function for $(t_n)_{n\geq 0}$ is $B(A(x))$ which is equal to

$$e^{-\frac{x^2}{4} + \frac{x}{2-2x} - \ln(\sqrt{1-x})} = \frac{e^{-\frac{x^2}{4} + \frac{x}{2-2x}}}{\sqrt{1-x}}.$$

□

Problem 9 *Using generating functions, prove that the number of partitions of n into distinct parts equals the number of partitions of n where each part is odd.*

Solution. Let a_n be the number of partitions of n into distinct parts and suppose $A(x) = \sum_{i=0}^{\infty} a_i x^i$ is the generating function for a_n. As an integer may appear at most once in a partition of n, $A(x)$ must be composed of only factors in the form $(1 + x^k)$ for all positive integers k. Therefore,

$$A(x) = \prod_{k=1}^{\infty} (1 + x^k) = \prod_{k=1}^{\infty} \frac{1 - x^{2k}}{1 - x^k} = \prod_{k \text{ odd}} \frac{1}{1 - x^k}.$$

However, the last product may be represented as

$$\prod_{k \text{ odd}} (1 + x^k + x^{2k} + \cdots).$$

This representations tells us that we can make partitions out of any number of odd integers, where the term x^{ak} represents using k a total of a times in the partitions. This means that $A(x)$ is the generating function for the number of partitions into distinct parts in addition to the number of partitions into odd parts, implying they are equal.

$\square$